

# Multiple Dedekind Zeta Functions

Ivan Horozov \*

*Department of Mathematics,  
Washington University in St. Louis,  
One Brookings Drive,  
St Louis, MO 63130, USA*

## Abstract

In this paper we define multiple Dedekind zeta values (MDZV), using a new type of iterated integrals, called iterated integrals on a membrane. One should consider them as an number theoretic analogue of multiple zeta values. Over imaginary quadratic fields MDZV capture in particular multiple Eisenstein series [ZGK]. We give an analogue of multiple Eisenstein series over real quadratic field. And an alternative definition of values of multiple Eisenstein-Kronecker series [G2]. Each of them as a special case of multiple Dedekind zeta values. MDZV are interpolated into functions that we call multiple Dedekind zeta functions (MDZF). We show that MDZF have integral representation and can be written as infinite sums.

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\*E-mail: last name@math.wustl.edu

## 0 Introduction

Multiple Dedekind zeta functions generalize Dedekind zeta functions in the same way as multiple zeta functions generalize Riemann zeta function. Let us recall known definitions of the above functions. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s},$$

where  $n$  is an integer. Multiple zeta functions are defined as

$$\zeta(s_1, \dots, s_d) = \sum_{0 < n_1 < \dots < n_d} \frac{1}{n_1^{s_1} \dots n_d^{s_d}},$$

where  $n_1, \dots, n_d$  are integers. Both functions  $\zeta(s)$  and  $\zeta(s_1, \dots, s_d)$  were defined by Euler [Eu]. The Riemann zeta function is closely related to the ring of integers.

Dedekind zeta function  $\zeta_K(s)$  is an analogue of the Riemann zeta function, which is closely related to the algebraic integers  $\mathcal{O}_K$  in a number field  $K$ . It is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where the sum is over all ideals  $\mathfrak{a}$  different from the zero ideal  $(0)$  and  $N(\mathfrak{a}) = \#|\mathcal{O}_K/\mathfrak{a}|$  is the norm of the ideal  $\mathfrak{a}$ .

A definition of multiple Dedekind zeta functions should combine ideas from multiple zeta functions and from Dedekind zeta functions.

Let us recall the definition of Masri [Mas]. Let  $K_1, \dots, K_d$  be number fields and let  $\mathcal{O}_{K_i}$ , for  $i = 1, \dots, d$  be the corresponding rings of integers. Let  $\mathfrak{a}_i$ , for  $i = 1, \dots, d$  be ideals in  $\mathcal{O}_{K_i}$ , respectively. Then he defines

$$\zeta(K_1, \dots, K_d; s_1, \dots, s_d) = \sum_{0 < N(\mathfrak{a}_1) < \dots < N(\mathfrak{a}_d)} \frac{1}{N(\mathfrak{a}_1)^{s_1} \dots N(\mathfrak{a}_d)^{s_d}}.$$

We propose a different definition. The advantage of our definition is that it leads to more properties: analytic, topological and algebra-geometric. Let us give an explicit formula for a multiple Dedekind zeta function, in a case when it is easier to formulate. Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $U_K$  be the group of units in  $\mathcal{O}_K$ . Let  $C$  be a cone inside a fundamental domain of  $\mathcal{O}_K$  modulo  $U_K$ . (More precisely,  $C$  has to be an unimodular simple cone as defined in Section 3. A fundamental domain for  $\mathcal{O}_K$  modulo  $U_K$  can be written as a finite union of unimodular simple cones.) For such a cone  $C$ , we define a multiple Dedekind zeta function

$$\zeta_K(s_1, \dots, s_d) = \sum_{\alpha_1, \dots, \alpha_d \in C} \frac{1}{N(\alpha_1)^{s_1} N(\alpha_1 + \alpha_2)^{s_2} \dots N(\alpha_1 + \dots + \alpha_d)^{s_d}}.$$

The key new ingredient in the definition of multiple Dedekind zeta functions is the definition of iterated integrals on a membrane. This is an analogue of iterated path integrals to higher dimensions. In the iterated integrals on a membrane the iteration happens in  $n$ -directions. The first place, where such iterated integrals were defined

is in [H] which generalizes Manin's non-commutative modular symbol [Man] to higher dimensions in some cases, essentially for Hilbert modular surfaces.

### Structure of the paper:

In Section 2, we give two Definitions of iterated integrals on a membrane. The second Definition is the one needed for the definition of multiple Dedekind zeta values.

In Section 3, we use some basic algebraic number theory (see [IR]), in order to construct the functions that we integrate. We use an idea of Shintani (see [Sh], [C]) for defining a cone, which is a fundamental domain for the algebraic integers  $\mathcal{O}_K$  in a number field  $K$  modulo the units  $U_K$ . More precisely, we define an unimodular cone (Definition 3.1) and a simple cone (Definition 3.2). We associate a product of geometric series to every unimodular simple cone. This is the type of functions that we integrate. Lemma 3.7 shows that a fundamental domain for the non-zero integer  $\mathcal{O}_K - \{0\}$  modulo the units  $U_K$  can be written as a finite union of unimodular simple cones.

In Section 4, we define Dedekind polylogarithms associated to an unimodular simple cone. Theorem 4.2 expresses Dedekind zeta values in terms of Dedekind polylogarithms. The heart of the section is Definition 4.4 of multiple Dedekind zeta values and Definition 4.6 of multiple Dedekind zeta functions.

At the end of the Section there are many examples. Most noticeable are Examples 1 and 2 of the simplest multiple Dedekind zeta values, Example 3, expressing partial Eisenstein-Kronecker series associated to an imaginary quadratic ring as multiple Dedekind values (see [G2], section 8.1) Example 4 considers multiple Eisenstein series (for an alternative definition see [G2], Section 8.2). Examples 5 and 6 are the simplest multiple Dedekind zeta functions. Example 7 is a mixture of Eisenstein series and Dedekind zeta function. Example 9 gives the formula for a general multiple Dedekind zeta function.

In Section 5, we prove an analytic continuation of multiple Dedekind zeta functions, which allows us to consider special values of multiple Eisenstein series (see [ZGK]) as values of multiple Dedekind zeta functions. Examples 10 and 11, illustrate this relation. And example 12 is an analogue of values Eisenstein series associated to a real quadratic field.

In Section 6, we state a conjecture about periods and multiple Dedekind zeta values.

## 1 Examples

We are going to present several examples of Riemann zeta values and multiple zeta values in order to introduce a key example of multiple Dedekind zeta value as an iterated integral. In stead of considering a general number field, which we will do in the later sections, we will examine only the ring of Gaussian integers. Also we will ignore questions about convergence. Such questions will be addressed in Sections 3.

Let us recall the  $n$ -th polylogarithms and their relation to Riemann zeta values. Let us define the first polylogarithm

$$Li_1(x_1) = \int_0^{x_1} \frac{dx_0}{1-x_0} = \int_0^{x_1} (x_0 + x_0^2 + x_0^3 + \dots) \frac{dx_0}{x_0} = x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \dots$$

The second polylogarithm is

$$Li_2(x_2) = \int_0^{x_2} Li_1(x_1) \frac{dx_1}{x_1} = x_2 + \frac{x_2^2}{2^2} + \frac{x_2^3}{3^2} + \dots$$

Note that  $\zeta(2) = Li_2(1)$ . The  $n$ -th polylogarithm is defined as

$$Li_n(x_n) = \int_0^{x_n} Li_{n-1}(x_{n-1}) \frac{dx_{n-1}}{x_{n-1}}. \quad (1.1)$$

By a direct computation it follows that

$$Li_n(x) = x + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \dots$$

The following relation is straightforward:

$$\zeta(n) = Li_n(1).$$

Using Equation 1.1, we can express the  $n$ -th polylogarithm as

$$Li_n(x_n) = \int_{0 < x_0 < x_1 < \dots < x_n} \frac{dx_0}{1 - x_0} \wedge \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_{n-1}}{x_{n-1}}.$$

This is a presentation of the  $n$ -th polylogarithm as an iterated integral. We will give also an example of a multiple polylogarithm and a multiple zeta value.

Let  $x_i = e^{-t_i}$ . Then the  $n$ -th polylogarithm can be written as an iterated integral in the variables  $t_0, \dots, t_n$  in the following way

$$Li_n(e^{-t_n}) = \int_{t_0 > t_1 > \dots > t_n} \frac{dt_0 \wedge \dots \wedge dt_{n-1}}{e^{t_0} - 1}.$$

As an infinite sum, we have

$$Li_n(e^{-t}) = \sum_{k > 0} \frac{e^{-kt}}{k^n}. \quad (1.2)$$

Let us define

$$Li_{1,1}(1, x_2) = \int_0^{x_2} Li_1(x_1) \frac{dx_1}{1 - x_1} = \int_0^{x_2} \left( \sum_{m=1}^{\infty} \frac{x_1^m}{m} \right) \left( \sum_{n=1}^{\infty} x_1^n \right) \frac{dx_1}{x_1} = \sum_{m,n=1}^{\infty} \frac{x_2^{m+n}}{m(m+n)}.$$

Let  $x_i = e^{-t_i}$ . Then the  $Li_{1,1}(1, e^{-t_2})$  can be written as an iterated integral in the variables  $t_0, t_1, t_2$  in the following way

$$Li_{1,1}(1, e^{-t_2}) = \int_{t_0 > t_1 > t_2} \frac{dt_0 \wedge dt_1}{(e^{t_0} - 1)(e^{t_1} - 1)}.$$

As an infinite sum, we have

$$Li_{1,1}(1, e^{-t}) = \sum_{k < l} \frac{e^{-(k+l)t}}{k(k+l)}. \quad (1.3)$$

An example of a multiple zeta value is

$$\zeta(1, 2) = \sum_{m,n=1}^{\infty} \frac{1}{m(m+n)^2} = \int_0^1 Li_{1,1}(x_2) \frac{dx_2}{x_2}.$$

We generalize the Equations 1.2 and 1.3 to Gaussian integers. Let

$$Cone = \mathbb{N}\{1, i\} = \{\alpha \in \mathbb{Z}[i] | \alpha = a + ib, a, b \in \mathbb{N}\}.$$

Let

$$f_0(Cone; t_1, t_2) = \sum_{\alpha \in Cone} \exp(-\alpha t_1 - \bar{\alpha} t_2).$$

$$f_1(Cone, u_1, u_2) = \int_{u_1}^{\infty} \int_{u_2}^{\infty} f_0(Cone; t_1, t_2) dt_1 \wedge dt_2.$$

**Lemma 1.1** (a)  $\int_u^{\infty} e^{-kt} dt = \frac{e^{-ku}}{k}$ ;

(b) Let  $N(\alpha) = \alpha \bar{\alpha}$ . Then

$$\int_{u_1}^{\infty} \int_{u_2}^{\infty} \exp(-\alpha t_1 - \bar{\alpha} t_2) dt_1 \wedge dt_2 = \frac{\exp(-\alpha u_1 - \bar{\alpha} u_2)}{N(\alpha)}.$$

*Proof.* left for the reader. Then

$$f_1(Cone; u_1, u_2) = \frac{\sum_{\alpha \in Cone} \exp(-\alpha u_1 - \bar{\alpha} u_2)}{N(\alpha)}$$

Similarly, we define

$$f_n(Cone; u_1, u_2) = \int_{u_1}^{\infty} \int_{u_2}^{\infty} f_{n-1}(Cone; t_1, t_2) dt_1 \wedge dt_2.$$

Then

$$f_n(Cone; u_1, u_2) = \frac{\sum_{\alpha \in Cone} \exp(-\alpha u_1 - \bar{\alpha} u_2)}{N(\alpha)^n}.$$

Note that

$$f_n(Cone; 0, 0) = \left( \sum_{(\alpha) \subset \mathbb{Z}[i]} - \sum_{\alpha \in \mathbb{N}} \right) \frac{1}{N((\alpha))^n} = \zeta_{\mathbb{Q}(i)}(n) - \zeta(2n),$$

where  $\zeta_{\mathbb{Q}(i)}(n)$  is a Dedekind zeta value and  $\zeta(2n)$  is a Riemann zeta value. Now we can define a number field analogue of  $Li_{1,1}(1, \epsilon^{-t})$  as

$$f_{1,1}(Cone; v_1, v_2) = \int_{v_1}^{\infty} \int_{v_2}^{\infty} f_1(Cone; u_1, u_2) f_0(Cone; u_1, u_2) du_1 \wedge du_2.$$

**Lemma 1.2**

$$f_{1,1}(Cone; v_1, v_2) = \sum_{\alpha, \beta \in Cone} \frac{\exp(-(\alpha + \beta)v_1 - (\bar{\alpha} + \bar{\beta})v_2)}{N(\alpha)N(\alpha + \beta)}.$$

*Proof.*

$$f_{1,1}(Cone; v_1, v_2) = \int_{v_1}^{\infty} \int_{v_2}^{\infty} f_1(Cone; u_1, u_2) f_0(Cone; u_1, u_2) du_1 \wedge du_2 = \quad (1.4)$$

$$= \int_{v_1}^{\infty} \int_{v_2}^{\infty} \sum_{\alpha \in Cone} \frac{\exp(-\alpha u_1 - \bar{\alpha} u_2)}{N(\alpha)} \sum_{\beta \in Cone} \exp(-\beta u_1 - \bar{\beta} u_2) du_1 \wedge du_2 = \quad (1.5)$$

$$= \int_{v_1}^{\infty} \int_{v_2}^{\infty} \sum_{\alpha, \beta \in Cone} \frac{\exp(-(\alpha + \beta) v_1 - (\bar{\alpha} + \bar{\beta}) v_2)}{N(\alpha)} du_1 \wedge du_2 = \quad (1.6)$$

$$= \sum_{\alpha, \beta \in Cone} \frac{\exp(-(\alpha + \beta) v_1 - (\bar{\alpha} + \bar{\beta}) v_2)}{N(\alpha) N(\alpha + \beta)}. \quad (1.7)$$

Similalry, we define

$$f_{1,2}(Cone, w_1, w_2) = \int_{w_1}^{\infty} \int_{w_2}^{\infty} f_{1,1}(Cone; v_1, v_2) dv_1 \wedge dv_2.$$

A direct computation leads to

$$f_{1,2}(Cone; w_1, w_2) = \sum_{\alpha, \beta \in Cone} \frac{\exp(-(\alpha + \beta) w_1 - (\bar{\alpha} + \bar{\beta}) w_2)}{N(\alpha) N(\alpha + \beta)^2}.$$

We define a multiple Dedekind zeta value as

$$\zeta_{\mathbb{Q}(i)}^{Cone}(1, 2) = f_{1,2}(Cone; 0, 0) = \sum_{\alpha, \beta \in Cone} \frac{1}{N(\alpha) N(\alpha + \beta)^2}.$$

Now let us give a relation of Dedekind zeta values and multiple Dedekind zeta values to iterated integrals. Note that

$$f_2(Cone; v_1, v_2) = \int_{t_1 > u_1 > v_1; t_2 > u_2 > v_2} f_0(Cone; t_1, t_2) dt_1 \wedge dt_2 \wedge du_1 \wedge du_2.$$

Also

$$f_{1,1}(Cone; v_1, v_2) = \int_{t_1 > u_1 > v_1; t_2 > u_2 > v_2} f_0(Cone; t_1, t_2) dt_1 \wedge dt_2 \wedge f_0(Cone; u_1, u_2) du_1 \wedge du_2$$

and if we put  $C = Cone$  then

$$f_{1,2}(C; w_1, w_2) = \quad (1.8)$$

$$= \int_{t_1 > u_1 > v_1 > w_1; t_2 > u_2 > v_2 > w_2} f_0(C; t_1, t_2) dt_1 \wedge dt_2 \wedge f_0(C; u_1, u_2) du_1 \wedge du_2 \wedge dv_1 \wedge dv_2 \quad (1.9)$$

## 2 Iterated integrals on a membrane

For each  $i$ ,  $i = 1, \dots, n$ , we cut the admissible interval for  $t_i$  into subintervals. For fixed  $i$ , let

$$t_i^1 > t_i^2 > \dots > t_i^m$$

values of  $t_i$  in each subinterval.

Consider variables by  $t_i^j$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . The subscript  $i$  corresponds to  $i$ -th direction.

Let  $D$  be a domain defined in terms of the variables  $t_i^j$ , by

$$D = \{(t_i^j) | t_i^1 > t_i^2 > \dots > t_i^m > 0\}.$$

We associate differential  $n$ -forms  $\omega_j$  on a manifold  $M$ , for  $j = 1, 2, \dots, m$ . Let

$$g : (0, \infty)^n \rightarrow M$$

be a smooth map. We will call such a map a *membrane*.

**Definition 2.1** *An iterated integral on a membrane  $g$ , in terms of  $n$ -forms  $\omega_j$ ,  $j = 1, \dots, m$ , is defined as*

$$\int_g \omega_1 \dots \omega_m = \int_D \bigwedge_{j=1}^m g^* \omega_j(t_1^j, \dots, t_n^j).$$

**Definition 2.2** *A shuffle between two ordered sets*

$$S_1 = \{1, \dots, p\}$$

and

$$S_2 = \{p+1, \dots, p+q\}$$

is a permutation  $\tau$  of the union  $S_1 \cup S_2$ , such that

1. for  $a, b \in S_1$ , we have  $\tau(a) < \tau(b)$  if  $a < b$ ;
2. for  $a, b \in S_2$ , we have  $\tau(a) < \tau(b)$  if  $a < b$ ;

We denote the set of shuffles between two ordered sets of orders  $p$  and  $q$ , respectively, by  $Sh(p, q)$ .

The definition of an iterated integral on a membrane is associated with the following objects:

1.  $g : (0, \infty)^n \rightarrow \mathbb{C}^n$ , a smooth map, which maps the  $i$ -th real coordinate to the  $i$ -th complex coordinate;
2.  $\omega_1, \dots, \omega_m$  differential  $n$ -forms on  $\mathbb{C}^n$ ;
3.  $\alpha_i = dz_i$  differential 1-forms on  $\mathbb{C}^n$ , for  $i = 1, \dots, n$ ;
4. for each  $i \in \{1, \dots, n\}$ , take  $m_i$  copies of the 1-form  $\alpha_i$  and a shuffle  $\tau_i \in Sh(m, m_i)$ ;
5.  $\tau = (\tau_1, \dots, \tau_n)$ , the set of  $n$  shuffles  $\tau_1, \dots, \tau_n$ .

**Definition 2.3** Given the above data, we define an iterated integral on a membrane  $g$ , involving  $n$ -forms and 1-forms, as

$$\begin{aligned} & \int_{(g,\tau)} \omega_1 \dots \omega_m (dz_1)^{m_1} \dots (dz_n)^{m_n} = \\ & = \int_D \bigwedge_{j=1}^m g^* \omega_j(t_1^{\tau_1(j)}, \dots, t_n^{\tau_n(j)}) \bigwedge_{i=1}^n \bigwedge_{j=m+1}^{m+m_i} g^* \alpha_i(t_i^{\tau_i(j)}), \end{aligned} \quad (2.10)$$

where  $t_i^j$ 's belong to the domain

$$D = \{t_i^j | t_i^1 > t_i^2 > \dots > t_i^{m+m_i} > 0\}.$$

Note that  $g^* \alpha_i(t_i^{\tau_i(j)}) = dt_i^{\tau_i(j)}$ .

### 3 Cones and geometric series

**Definition 3.1** We call  $C \subset \mathcal{O}_K$  a unimodular cone if

(1)  $C$  has exactly  $m$  edges,  $m \leq n$ , such that the generators of the edges generate the whole cone, where  $n = [K : \mathbb{Q}]$ . If  $e_1, \dots, e_m$  are the generators of the edges, then

$$C = \mathbb{N}\{e_1, \dots, e_m\} = \{\alpha \in \mathfrak{a} | \alpha = \sum_{i=1}^m a_i e_i, a_i \in \mathbb{N}\},$$

(2) If  $u \neq 1$  is a unit in  $K$  and  $\alpha \in C$  then  $u\alpha \notin C$ .

**Definition 3.2** A simple cone is a cone  $C$  such that for any two elements  $\alpha$  and  $\beta$  in  $C$  and any embedding  $\sigma : K \rightarrow \mathbb{C}$ , we have

$$\arg(\sigma(\alpha)) \neq -\arg(\sigma(\beta)).$$

**Definition 3.3** Let  $C = \mathbb{N}\{e_1, \dots, e_m\}$  be a simple unimodular cone with generators  $e_j$  for  $j = 1, \dots, m$  and  $m \leq n = [K : \mathbb{Q}]$ . We define

$$f_0(C, t_1, \dots, t_n) = \sum_{\alpha \in C} \exp(-\sum_{i=1}^n \sigma_i(\alpha) t_i),$$

where  $t_i$  belongs to a sector  $S_i(C)$  in the complex plane  $\mathbb{C}$ , defined by the conditions

$$S_i(C) = \{t_i \in \mathbb{C} | \operatorname{Re}(\sigma_i(e_j) t_i) > 0, \text{ for } j = 1, \dots, m\}.$$

Note that  $S_i(C)$  is non-empty, when the cone  $C$  is simple.

**Lemma 3.4** The function  $f_0$  is uniformly convergent for  $t_i$  in compact subsets of  $S_i$  for  $i = 1, \dots, n$ .



*Proof.* For  $t_i$  in compact subset of  $S_i$ , we have that  $\operatorname{Re}(\sigma_i(e_j)t_i) > 0$ . Let

$$y_j = \prod_{i=1}^n \exp(-\sigma_i(e_j)t_i).$$

Then  $|y_j| < 1$  and

$$f_0(C, t_1, \dots, t_n) = \prod_{j=1}^m \frac{y_j}{1 - y_j}. \quad (3.11)$$

□

**Corollary 3.5** *The function  $f_0(C, t_1, \dots, t_n)$  has analytic continuation to all values of  $t_1, \dots, t_n$ , except at*

$$\sum_{i=1}^n \sigma_i(e_j)t_i \in 2\pi i\mathbb{Z},$$

for  $j = 1, \dots, m$ .

*Proof.* Use the product formula 3.11 in terms of geometric series in  $y_j$ . The right hand side of 3.11 makes sense for all  $y_j \neq 1$ , which gives the conditions in this corollary. □

**Definition 3.6** *We define Cone as a fundamental domain of*

$$\mathcal{O}_K - \{0\} \bmod U_K.$$

For an ideal  $\mathfrak{a}$ , let

$$\operatorname{Cone}(\mathfrak{a}) = \operatorname{Cone} \cap \mathfrak{a}.$$

**Lemma 3.7** *For any ideal  $\mathfrak{a}$  the cone  $\operatorname{Cone}(\mathfrak{a})$  can be written as a finite union of unimodular simple cones.*

*Proof.* It is a simple observation that  $\operatorname{Cone}(\mathfrak{a})$  can be written as a finite union of unimodular cones. We have to show that we can subdivide each of the unimodular cones into finite union of unimodular simple cones. Consider the domain

$$T = \prod_{\nu \in \infty} \frac{x_\nu}{|x_\nu|},$$

where the product is taken over the infinite places of the ideles in the number field  $K$ . Then  $T$  is a finite union of compact tori.

If  $C$  is an unimodular cone, consider the closure of its projection in  $T$ . Denote it by  $\bar{C}$ . Then one can cut the cone  $C$  into finitely many simple cones such that for  $C_i$  and any embedding  $\sigma$  of  $K$  into  $\mathbb{C}$ , we have that  $\arg(\sigma(\alpha)) \neq -\arg(\sigma(\beta))$ , for  $\alpha, \beta \in C_i$ . Then we can cut each  $C_i$  into finitely many unimodular cones  $C_{ij}$ . Since  $C_{ij}$  is a subcone of a simple cone, it is a simple cone as well. Thus, the cones  $C_{ij}$ 's are finitely many simple unimodular cones, whose union gives  $\operatorname{Cone}(\mathfrak{a})$ . □

**Lemma 3.8** *Let  $C$  be an unimodular simple cone. Then for every  $\alpha \in C$ , we have*

$$N_{K/\mathbb{Q}}((\alpha)) = \epsilon(C)N_{K/\mathbb{Q}}(\alpha),$$

where  $\epsilon(C) = \pm 1$  depends only on the cone  $C$ .

*Proof.* Note that on the left we have a norm of an ideal and on the right we have a norm of a number. Since  $C$  is a simple cone, we have that for all real embeddings  $\sigma : K \rightarrow \mathbb{R}$ , the signs of  $\sigma(\alpha)$  and  $\sigma(\beta)$  are the same for all  $\alpha$  and  $\beta$  in  $C$ . Let  $\epsilon_\sigma$  be the sign of  $\sigma(\alpha)$  for each real embedding  $\sigma$ . Then the product over all real embeddings of  $\epsilon_\sigma$  gives  $\epsilon(C)$ .  $\square$

## 4 Definition of multiple Dedekind zeta functions

### 4.1 Dedekind zeta functions and cones

Let  $n = [K : \mathbb{Q}]$  be the degree of the number field  $K$  over  $\mathbb{Q}$ . Let  $\mathcal{O}_K$  be the ring of integers in  $K$ . And let  $U_K$  be the group of units in  $K$ . We are going to use an idea of Shintani [C] by examining Dedekind zeta functions in terms of a cone inside the ring of integers.

Let us recall the Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where  $\mathfrak{a}$  is an ideal in  $\mathcal{O}_K$ . We also define a partial Dedekind zeta function by summing over ideals in a given ideal class  $[\mathfrak{a}]$

$$\zeta_{K,[\mathfrak{a}]}(s) = \sum_{\mathfrak{b} \in [\mathfrak{a}]} N_{K/\mathbb{Q}}(\mathfrak{b})^{-s},$$

Now let us consider a partial Dedekind zeta functions  $\zeta_{K,[\mathfrak{a}]^{-1}}(s)$ , corresponding to an ideal class  $[\mathfrak{a}]^{-1}$ , where  $\mathfrak{a}$  is an integral ideal. For every integral ideal  $\mathfrak{b}$  in the class  $[\mathfrak{a}]^{-1}$ , we have that

$$\mathfrak{a}\mathfrak{b} = (\alpha),$$

where  $\alpha \in \mathfrak{a}$ . Then

$$N_{K/\mathbb{Q}}(\mathfrak{b}) = N_{K/\mathbb{Q}}(\mathfrak{a})^{-1} N_{K/\mathbb{Q}}((\alpha)).$$

Let

$$Cone(\mathfrak{a}) = \bigcup_{i=1}^{n(\mathfrak{a})} C_i(\mathfrak{a}),$$

where  $n(\mathfrak{a})$  is a positive integer and  $C_i(\mathfrak{a})$ 's are unimodular simple cones.

Then,

$$\begin{aligned} \zeta_{K,[\mathfrak{a}]^{-1}}(s) &= \sum_{\mathfrak{b} \in [\mathfrak{a}]^{-1}} N_{K/\mathbb{Q}}(\mathfrak{b})^{-s} = \\ &= N_{K/\mathbb{Q}}(\mathfrak{a})^s \sum_{i=1}^{n(\mathfrak{a})} \epsilon(C_i(\mathfrak{a}))^s \sum_{\alpha \in C_i(\mathfrak{a})} N_{K/\mathbb{Q}}(\alpha)^{-s}, \end{aligned} \tag{4.12}$$

where  $\epsilon(C_i(\mathfrak{a})) = \pm 1$ , depending on the cone.

## 4.2 Dedekind polylogarithms

We are going to give an example of higher dimensional iteration in order to illustrate the usefulness of this procedure. For a simple unimodular cone  $C$ , we define

$$f_m(C, u_1, \dots, u_n) = \int_{\infty}^{t_1} \dots \int_{\infty}^{t_n} f_{m-1}(C, t_1, \dots, t_n) du_1 \wedge \dots \wedge du_n,$$

where  $|t_i| \in (|u_i|, \infty)$  and  $\arg(u_i) = \arg(t_i)$ . This is an iteration, giving the simplest type of iterated integrals on a membrane. We start the induction on  $m$  from  $m = 0$ . Recall that  $f_0$  was defined in terms of a cone in Definition 3.3.

As an infinite sum, we can express  $f_m$  as

$$f_m(C, t_1, \dots, t_n) = \sum_{\alpha \in C} \frac{\exp(-\sum_{i=1}^n \sigma_i(\alpha) t_i)}{N_{K/\mathbb{Q}}(\alpha)^m}.$$

Note that

$$N_{K/\mathbb{Q}}(\alpha) = \sigma_1(\alpha) \dots \sigma_n(\alpha).$$

**Definition 4.1** We define an  $m$ -th Dedekind polylogarithm, associated to a number field  $K$  and an unimodular simple cone  $C$ , to be

$$Li_m^K(C, X_1, \dots, X_n) = f_m(C, -\log(X_1), \dots, -\log(X_n)).$$

**Theorem 4.2** Dedekind zeta value at  $s = m > 1$  can be written as a finite  $\mathbb{Q}$ -linear combination of Dedekind polylogarithms evaluated at  $(X_1, \dots, X_n) = (1, \dots, 1)$ .

*Proof.* If  $\mathfrak{a}_1, \dots, \mathfrak{a}_h$  be integral ideals in  $\mathcal{O}_K$ , representing all the ideal classes, then

$$\zeta_K(m) = \sum_{j=1}^h N_{K/\mathbb{Q}}(\mathfrak{a}_j)^m \sum_{i=1}^{n(\mathfrak{a}_j)} \epsilon(C_i(\mathfrak{a}_j))^m f_m(C_i(\mathfrak{a}_j), 0, \dots, 0),$$

where  $C_i(\mathfrak{a}_j)$  are simple unimodular cones such that

$$\bigcup_{i=1}^{n(\mathfrak{a}_j)} C_i(\mathfrak{a}_j) = \text{Cone}(\mathfrak{a}_j)$$

and  $\epsilon(C_i(\mathfrak{a}_j)) = \pm 1$ , depending on the cone. Where are the iterated integrals? Consider Definition 1.1 with trivial permutations  $\rho_1, \dots, \rho_n$  and with differential forms

$$\omega_1 = f_0(C_i(\mathfrak{a}_j), t_1, \dots, t_n) dt_1 \wedge \dots \wedge dt_n,$$

$$\omega_2 = \omega_3 = \dots = \omega_m = dt_1 \wedge \dots \wedge dt_n.$$

And let  $g$  be the identity map on  $(0, \infty)^n$ . Then the corresponding iterated integral on a membrane gives

$$f_m(C_i(\mathfrak{a}_j), t_1, \dots, t_n).$$

□

### 4.3 Multiple Dedekind zeta values

We are going to define a shuffle of ordered sets. We are going to use  $n$  such shuffles in the definition of multiple Dedekind zeta values at the positive integers.

Let  $m_1, \dots, m_n$  be positive integers. The positive integer  $m_i$  will denote the number of times the differential form  $dt_i$  occurs. We define the following ordered sets:

$$S = \{1, 2, \dots, m\},$$

$$S_i = \{m + 1, \dots, m + m_i\},$$

**Definition 4.3** Denote by  $Sh^1(p, q)$  the subset of all shuffles  $\tau \in Sh(p, q)$  of the two sets  $\{1, \dots, p\}$  and  $\{p + 1, \dots, p + q\}$  such that  $\tau(1) = 1$

For the definition of multiple Dedekind zeta values at the positive integers, we use Definition 2.3, where we take the  $n$ -forms to be

$$\omega_j = f_0(C_j, z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n,$$

for  $j = 1, \dots, m$ , and for unimodular simple cones  $C_1, \dots, C_m$ ; and the 1-forms to be  $dz_1, \dots, dz_n$  on  $\mathbb{C}^n$ .

**Definition 4.4** (Multiple Dedekind zeta values) For fixed  $i$ , we have  $\tau_i \in Sh^1(m, m_i)$ . Then  $\tau_i$  acts on  $S \cup S_i$ . The image of  $S$  under the action of  $\tau_i$  cuts the ordered set  $S \cup S_j$  into subintervals. We denote the length of each such closed subinterval minus 1 by  $k_{ij}$ , where the intervals in increasing order of the elements have lengths  $k_{i1}, k_{i2}, \dots, k_{im}$ . We define multiple Dedekind zeta values at the positive integers by

$$\zeta_{K, C_1, \dots, C_m}(\rho, k_{11}, \dots, k_{nm}) = \int_{(g, \tau)} \omega_1 \dots \omega_m (dz_1)^{m_1} \dots (dz_n)^{m_n}$$

**Examples:** 1. Let  $C$  be an unimodular simple cone. If all values  $k_{i,1}$  are equal to  $k$ , then

$$\zeta_{K, C}(k, \dots, k) = \sum_{\alpha \in C} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^k}.$$

2. Let

$$k = k_{1,1} = \dots = k_{n,1}$$

and

$$l = k_{1,2} = \dots = k_{n,2}$$

be positive integers greater than 1. Finally, let  $C_1$  and  $C_2$  be unimodular simple cones in the ring of integers  $\mathcal{O}_K$  of a number field  $K$ . Then the corresponding multiple Dedekind zeta value can be written both as a sum and as an integral:

$$\zeta_{K, C_1, C_2}(k, \dots, k, l, \dots, l) = \sum_{\alpha \in C_1, \beta \in C_2} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^k N_{K/\mathbb{Q}}(\alpha + \beta)^l} \quad (4.13)$$

3. Let  $K$  be an imaginary quadratic field. Let  $C$  be an unimodular simple cone in  $\mathcal{O}_K$ . We can represent the cone  $C$  as an  $\mathbb{N}$ -module:  $C = \mathbb{N}\{\mu, \nu\}$ , for  $\mu, \nu \in \mathcal{O}_K$ . Put  $z = \mu/\nu$ . Consider Then

$$\zeta_{K,C}(k, k) = \sum_{\alpha \in C} \frac{1}{\alpha_1^k} = \nu^{-k} \sum_{a, b \in \mathbb{N}} \frac{1}{|az + b|^{2k}},$$

where the last sum is a portion of the  $k$ -th Eisenstein-Kronecker series. Similar series was considered in [G2], Section 8.1.

$$E_k(\tau) = \sum_{a, b \in \mathbb{Z}; (a, b) \neq (0, 0)} \frac{1}{|az + b|^{2k}}.$$

is an analogue of Eisenstein series.

4. With the notation of Example 3, consider  $\zeta_{K,C,C}(k_{1,1}, k_{2,1}, k_{1,2}, k_{2,2})$  with  $k_{1,1} = k_{2,1} = k$ ,  $k_{1,2} = k_{2,2} = l$ . Then, we obtain an analogue of values of multiple Eisenstein-Kronecker series

$$\zeta_{K,C,C}(k, k, l, l) = \nu^{-k-l} \sum_{a, b, c, d \in \mathbb{N}} \frac{1}{|az + b|^{2k} |(a+c)z + (b+d)|^{2l}}.$$

(See also examples 10 and 11 on page 18 of this paper.) An alternative generalization was considered in [G2], Section 8.2.

#### 4.4 Multiple Dedekind zeta functions

We will try to give some intuition behind the integral representation of the multiple zeta functions (see in [G1]). After that we will generalize the construction to define the number field analogues - multiple Dedekind zeta functions.

We will give two examples. One for  $\zeta(3)$  and one for  $\zeta(1, 2)$ . We have

$$\zeta(3) = \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3} = \tag{4.14}$$

$$= \int_{t_1 > t_2 > t_3 > 0} \frac{dt_1 \wedge dt_2 \wedge dt_3}{e^{t_1} - 1} = \tag{4.15}$$

$$= \int_0^\infty \frac{t^2 dt}{\Gamma(3)(e^{t_1} - 1)}. \tag{4.16}$$

The first equality is due to Kontsevich. It is examined in more details in Section 1. The second equality uses the change of variables  $x_i = e^{-t_i}$ . The last equality uses

**Lemma 4.5**

$$\int_{b > t_1 > t_2 > \dots > t_n > a} dt_1 \wedge dt_2 \wedge \dots \wedge dt_n = \frac{(b-a)^n}{\Gamma(n+1)}$$

Similarly,

$$\zeta(1, 2) = \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{1-t_2} \wedge \frac{dt_3}{t_3} = \tag{4.17}$$

$$= \int_{t_1 > t_2 > t_3 > 0} \frac{dt_1 \wedge dt_2 \wedge dt_3}{(e^{t_1} - 1)(e^{t_2} - 1)} = \quad (4.18)$$

$$= \int_{t_1 > t_2 > 0} \frac{dt_1}{(e^{t_1} - 1)} \wedge \frac{t_2^{2-1} dt_2}{\Gamma(1)\Gamma(2)(e^{t_2} - 1)} = \quad (4.19)$$

$$= \int_{(0, \infty)^2} \frac{u_1^{1-1} u_2^{2-1} du_1 \wedge du_2}{\Gamma(1)\Gamma(2)(e^{u_1+u_2} - 1)(e^{u_2} - 1)}. \quad (4.20)$$

The first two equalities are of the same type as in the previous example. For the third equality we use Lemma 4.5. For the last equality we use the change of variable

$$t_2 = u_2,$$

$$t_1 = u_1 + u_2,$$

where  $u_1 > 0$  and  $u_2 > 0$ . Following [G1], we can interpolate the multiple zeta values by

$$\zeta(s_1, \dots, s_d) = \Gamma(s_1)^{-1} \dots \Gamma(s_d)^{-1} \int_{(0, \infty)^d} \frac{u_1^{s_1-1} \dots u_d^{s_d-1} du_1 \wedge \dots \wedge du_d}{(e^{u_1+\dots+u_d} - 1)(e^{u_2+\dots+u_d} - 1) \dots (e^{u_d} - 1)}$$

If we denote by

$$f_0(\mathbb{N}; t) = \sum_{a \in \mathbb{N}} e^{-at},$$

then

$$f_0(\mathbb{N}, t) = \frac{1}{e^t - 1}$$

and

$$\zeta(s_1, \dots, s_d) = \Gamma(s_1)^{-1} \dots \Gamma(s_d)^{-1} \int_{(0, \infty)^d} \bigwedge_{j=1}^d f_0(\mathbb{N}; u_i + \dots + u_d) u_j^{s_j-1} du_j$$

Let  $n = [K : \mathbb{Q}]$  be the degree of the number field. We recall the definition of  $f_0$ ,

$$f_0(C; t_1, t_2, \dots, t_n) = \sum_{\alpha \in C} e^{-\sum_{i=1}^n \sigma_i(\alpha) t_i},$$

where  $\sigma_i : K \rightarrow \mathbb{C}$  run through all embeddings of the field  $K$  into the complex numbers. Let  $C_1$  and  $C_2$  be two unimodular simple cones. Then we define a multiple Dedekind zeta function as

$$\begin{aligned} \zeta_{K; C_1, C_2}(s_{1,1}, \dots, s_{n,2}) &= \Gamma(s_{1,1})^{-1} \dots \Gamma(s_{n,2})^{-1} \\ &\int_{(0, \infty)^{2n}} f_0(C_1; (u_{1,1} + u_{1,2}), \dots, (u_{n,1} + u_{n,2})) \times \\ &\times f_0(C_2; (u_{1,2}, \dots, u_{n,2})) \bigwedge_{i=1}^n u_{i,1}^{s_{i,1}-1} du_{i,1} \wedge \bigwedge_{i=1}^n u_{i,2}^{s_{i,2}-1} du_{i,2} \end{aligned} \quad (4.21)$$

This definition combines both double zeta function and multiple Dedekind zeta values with double iteration. More generally, we can interpolate all multiple Dedekind zeta values into multiple Dedekind zeta functions so that multiple zeta functions are particular cases.

**Definition 4.6** (*Multiple Dedekind zeta functions*) Let  $n = [K; \mathbb{Q}]$  be the degree of the number field. Let  $C_1, \dots, C_m$  be  $m$  unimodular simple cones in  $\mathcal{O}_K$ . Let  $u_i^j \in (0, \infty)$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . We define multiple Dedekind zeta functions by the integral

$$\zeta_{K; C_1, \dots, C_m}(s_{11}, \dots, s_{nm}) = \prod_{(i,j)=(1,1)}^{(n,m)} \Gamma(s_{ij})^{-1} \int_{(0,\infty)^{mn}} \bigwedge_{j=1}^m f_0(C_j; (u_{1,j} + \dots + u_{1,m}), \dots, (u_{n,j} + \dots + u_{n,m})) \bigwedge_{i=1}^n u_{i,j}^{s_{ij}-1} du_{i,j} \quad (4.22)$$

**Examples:**

5. If all variables  $s_{i,1}$ , for  $i = 1, \dots, n$  have the same value  $s$ , then

$$\zeta_{K,C}(s, \dots, s) = \sum_{\alpha \in C} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^s}.$$

6. In particular, if

$$s_j = s_{1,j} = s_{2,j} = \dots = s_{n,j}$$

for  $j = 1, 2$ , then

$$\zeta_{K,C_1,C_2}((1), \dots, (1), s_{1,1}, \dots, s_{n,2}) = \sum_{\alpha \in C_1, \beta \in C_2} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^{s_1} N_{K/\mathbb{Q}}(\alpha + \beta)^{s_2}}.$$

7. Let  $K$  be any number field, let  $m = 1$  and let  $C$  be an unimodular simple cone in  $\mathcal{O}_K$ . Then

$$\zeta_{K,C}(s_{1,1}, \dots, s_{n,1}) = \sum_{\alpha \in C} \frac{1}{\prod_{i=1}^n \alpha_i^{s_{i,1}}},$$

where  $\alpha_i = \sigma_i(\alpha)$  is the  $i$ -th embedding in the complex numbers.

8. Now, let  $m = 2$ . Then we have a double iteration. Let  $K$  be any number field. Let  $C_1$  and  $C_2$  be two unimodular simple cones. Then

$$\zeta_{K,C_1,C_2}(s_{1,1}, \dots, s_{n,2}) = \sum_{\alpha \in C_1, \beta \in C_2} \frac{1}{\prod_{i=1}^n \alpha_i^{s_{i,1}} (\alpha_i + \beta_i)^{s_{i,2}}}.$$

9. For the general form of a multiple Dedekind zeta function, we need: a number field  $K$ ; unimodular simple cones  $C_j$  in  $\mathcal{O}_K$ , for  $j = 1, \dots, m$ ; elements  $\alpha_{(j)} \in C_j$  for  $j = 1, \dots, m$ ; complex embeddings of the elements  $\alpha_{i,(j)} = \sigma_i(\alpha_{(j)})$ ; Then, in general, multiple Dedekind zeta function is

$$\begin{aligned} & \zeta_{K,C_1, \dots, C_m}(s_{1,1}, \dots, s_{n,m}) = \\ &= \sum_{\alpha_{(j)} \in C_j \text{ for } j=1, \dots, m} \frac{1}{\prod_{i=1}^n \prod_{j=1}^m \left( \sum_{k=1}^j \alpha_{i,k} \right)^{s_{i,j}}}. \end{aligned} \quad (4.23)$$

## 5 Analytic continuation of multiple Dedekind zeta functions

First, let us consider the function

$$f_0(C_j; t_1, \dots, t_n) = \sum_{\alpha \in C_j} e^{-\sum_{i=1}^n \alpha_i t_i},$$

where  $\alpha_i = \sigma_i(\alpha)$  and  $C_j$  is an unimodular simple cone. Let  $K_j$  be the rank of the cone  $C_j$ . Let  $\epsilon_{j,1}, \dots, \epsilon_{j,K_j}$  be the generators of the edges of the cone  $C_j$ . Let

$$v_{jk} = \sum_{i=1}^n \sigma_i(\epsilon_{j,k}) t_i$$

and

$$y_{j,k} = e^{-v_{j,k}}.$$

With the above notation for the unimodular simple cone  $C_j$ , we have a formula of the type of Equation 3.11

$$f_0(C_j; t_1, \dots, t_n) = \prod_{k=1}^{K_j} \frac{y_{j,k}}{1 - y_{j,k}}.$$

**Lemma 5.1** *In terms of  $t_1, \dots, t_n$ , the poles of  $f_0$  occur at the hyperplanes*

$$v_{j,k} = \sum_{i=1}^n \sigma_i(\epsilon_{j,k}) t_i = 2\pi i \mathbb{Z},$$

for  $k = 1, \dots, K_j$ , where the function has simple poles.

**Corollary 5.2** *The function*

$$\left( \prod_{k=1}^{K_j} v_{j,k} \right) f_0(C_j, t_1, \dots, t_n)$$

is homomorphic on  $\mathbb{C}^n$ .

**Theorem 5.3** *Multiple Dedekind zeta functions have meromorphic continuation to values of  $s_{ij} \in \mathbb{C}$ .*

*Proof.* We are going to prove meromorphic continuation of partial multiple Dedekind zeta functions. As a consequence, we obtain that multiple Dedekind zeta functions have meromorphic continuation, since they can be written as a finite sum of partial multiple Dedekind zeta functions.

Let

$$u_{i,j} = t_i^j - t_i^{j+1} \text{ for } j < m.$$

If  $j = m$ , we define  $u_{i,m} = t_i^m$ .



Let  $\{\epsilon_{j,k}\}_{k=1}^{K_j}$  be a basis for  $C_j$ . The cone  $C_j$  gives rise of geometric series with generators

$$v_{j,k} = \sum_{i=1}^n \sigma_i(\epsilon_{j,k}) t_i^j.$$

We can express  $v_{j,k}$  in terms of  $u_{i,j}$  by

$$v_{j,k} = \sum_{i=1}^n \sigma_i(\epsilon_{j,k}) (u_{i,j} + \dots + u_{i,m}).$$

Recall that the domain for  $u_{i,j}$  is  $(0, \infty)$ . Therefore, a multiple Dedekind zeta function can be written as

$$\int_{(0,\infty)^{mn}} \left( \prod_{j=1}^m \prod_{k=1}^{K_j} \frac{1}{(e^{v_{j,k}} - 1)} \right) \bigwedge_{(i,j)=(1,1)}^{(n,m)} \Gamma(s_{ij})^{-1} u_{i,j}^{s_{ij}-1} du_{i,j}.$$

Let

$$u_{i,j+} = \begin{cases} u_{i,j} & \text{for } u_{i,j} \geq 0 \\ 0 & \text{for } u_{i,j} < 0. \end{cases}$$

Let also

$$v_{j,k+} = \sum_{i=1}^n \sum_{j'=\sigma_i^{-1}(j)}^m \sigma_i(\epsilon_{jk}) x_{i,j'+}.$$

Now we define a function  $W(s_{ij}, s'_{ij})$  as the distribution

$$D = \frac{u_{i,j+}^{s_{ij}-1}}{\Gamma(s_{ij})} \frac{v_{j,k+}^{s'_{j,k}-1}}{\Gamma(s'_{j,k})} du_{i,j}$$

applied to the smooth function

$$\prod_{j=1}^m \prod_{k=1}^{K_j} \frac{v_{j,k}}{(e^{v_{j,k}} - 1)}.$$

From Gelfand-Shilov's approach [GSh], the function  $W$  has analytic continuation. We can obtain that first for the variables  $s_{i,j}$ . And for  $s'_{j,k}$ , after the following change of variables. Note that  $v_{j,k}$  and  $u_{i,j}$  are related by a linear change of variables. Let  $w_{i,j}$  be another basis among the variables  $u_{i,j}$  and  $v_{j,k}$ , including the variables  $v_{j,k}$ . Let also  $J$  be the Jacobian of the change of variables from  $u_{i,j}$  to  $w_{i,j}$ . Then

$$D = J \frac{u_{i,j+}^{s_{ij}-1}}{\Gamma(s_{ij})} \frac{v_{j,k+}^{s'_{j,k}-1}}{\Gamma(s'_{j,k})} dw_{i,j}.$$

We are interested in the function  $W(s_{ij}, s'_{j,k})$  at  $s'_{j,k} = 0$ .

**Lemma 5.4** *The function  $W(s_{ij}, s'_{j,k})$  has zeroes along the divisors  $s'_{j,k} = 0$ .*

*Proof.* The following function has meromorphic continuation

$$\begin{aligned} Z(s_{ij}, s'_{jk}) &= W(s_{ij}, s'_{jk}) \prod_{j=1}^m \prod_{k=1}^{K_j} s'_{jk}{}^{-1} = \\ &= \int_{(0,\infty)^{mn}} \prod_{j=1}^m \left( \prod_{k=1}^{K_j} \prod_{i=1}^n \frac{u_{i,j}^{s_{ij}-1}}{\Gamma(s_{ij})} \frac{v_{j,k}}{(e^{v_{j,k}} - 1)} \frac{v_{j,k}^{s'_{jk}-1}}{\Gamma(s'_{jk} + 1)} du_{i,j} \right) \end{aligned} \quad (5.24)$$

Moreover,  $Z(s_{ij}, 0)$  is multiple Dedekind zeta function, which is well defined for positive values  $s_{ij} > 1$ . Thus, the function  $Z$  has no poles along  $s'_{jk} = 0$ . Moreover,  $Z$  has meromorphic continuation. Therefore, the multiple Dedekind zeta functions have meromorphic continuation.

**Examples:** Using the analytic continuation, we can consider values of the multiple Dedekind zeta functions, when one or more of the arguments are zero, which allows us to express special values of multiple Eisenstein series (see [ZGK]) as multiple Dedekind zeta values.

10. Let  $K$  be an imaginary quadratic field. Let  $C$  be an unimodular simple cone in  $\mathcal{O}_K$ . We can represent the cone  $C$  as an  $\mathbb{N}$ -module:  $C = \mathbb{N}\{\mu, \nu\}$ , for  $\mu, \nu \in \mathcal{O}_K$ . Put  $z = \mu/\nu$ . Consider  $\zeta_{K,C}(k_{1,1}, k_{2,1})$  at  $k_{1,1} = k$  and  $k_{2,1} = 0$ .

Then

$$\zeta_{K,C}(k, 0) = \sum_{\alpha \in C} \frac{1}{\alpha_1^k} = \nu^{-k} \sum_{a,b \in \mathbb{N}} \frac{1}{(az + b)^k},$$

where the last sum is a portion of the  $k$ -th Eisenstein series.

$$E_k(\tau) = \sum_{a,b \in \mathbb{Z}; (a,b) \neq (0,0)} \frac{1}{(az + b)^k}.$$

is an analogue of Eisenstein series.

11. With the notation of Example 3, consider  $\zeta_{K,C,C}(k_{1,1}, k_{2,1}, k_{1,2}, k_{2,2})$  with  $k_{1,1} = k$ ,  $k_{1,2} = l$  and  $k_{2,1} = k_{2,2} = 0$ . Then, we obtain a value of multiple Eisenstein series

$$\zeta_{K,C,C}(k, 0, l, 0) = \nu^{-k-l} \sum_{a,b,c,d \in \mathbb{N}} \frac{1}{(az + b)^k ((a+c)z + (b+d))^l}.$$

12. Similarly, one can define analogue of values of the above Eisenstein series over real quadratic field  $K$ , by setting

$$E_k(z) = \nu^k \zeta_{K,C}(k, 0) = \sum_{\alpha \in C} \frac{1}{\sigma_1(\alpha)^k},$$

where  $C = \mathbb{N}\{\mu, \nu\} \subset \mathcal{O}_K$  is an unimodular simple cone in the ring of integers in a real quadratic field and  $\sigma_1 : K \rightarrow \mathbb{R}$  is one of the real embeddings.

For the correct definition in Example 12, we use the analytic continuation of multiple Dedekind zeta functions, Theorem 5.3. In fact, the infinite sum would converge for exactly one of the real embeddings and  $k > 2$ , we leave this as an exercise.

## 6 Final remarks

We do expect that multiple Dedekind zeta values should give periods in the sense of [KZ]. More precisely:

**Conjecture 6.1** *Let  $K$  be a number field. For any choice of unimodular simple cones  $C_1, \dots, C_m$ , in the ring of integers of a number field  $K$ , we have that the multiple Dedekind zeta values*

$$\zeta_{K; C_1, \dots, C_m}(k_{1,1}, \dots, k_{m,n})$$

*are periods, when the  $k_{1,1}, \dots, k_{m,n}$  are natural numbers greater than 1.*

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